Class field theory for curves over p-adic fields

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Abstract

We develop class field theory of curves over p-adic fields which extends the unramified theory of S. Saito [4]. The class groups which approximate abelian étale fundamental groups of such curves are introduced in the terms of algebraic K-groups by imitating G. Wiesend's class group for curves over finite fields [6].

1 Introduction

Let X be a regular curve over a finite field k with function field K, $\overline{X}$ the regular compactification of X, that is the regular and proper curve which contains X as an open subvariety, and X_∞ the finite set of closed points in the boundary $\overline{X} \smallsetminus X$ of X. Class field theory describes the abelian étale fundamental group $\pi_1(X)^{\mathrm{ab}}$ of X by a topological abelian group $\mathcal{C}_X$ which is called the class group. In terms of (Milnor) K-groups, the group $\mathcal{C}_X$ is the cokernel of the map

$$K_1(K) \to \bigoplus_{x \in X_0} K_0(k(x)) \oplus \bigoplus_{x \in X_\infty} K_1(K_x)$$

induced by the inclusion $K \hookrightarrow K_x$ and the boundary map $K_1(K_x) \to K_0(k(x))$, where $k(x)$ is the residue field at x, K_x is the completion of K at x and X_0 is the set of closed points in X (cf. [6]). The reciprocity map $\rho_X : \mathcal{C}_X \to \pi_1(X)^{\mathrm{ab}}$ is defined by class field theory of finite fields, local class field theory and the reciprocity law. It has dense image and the kernel is the connected component of 0 in $\mathcal{C}_X$.

The aim of this note is to develop class field theory for curves over *local fields*. Here, a local field means a complete discrete valuation field with

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finite residue field. Let X be a regular curve over a local field k with function field K. The *class group* $\mathcal{C}_X$ of X is defined to be the cokernel of the homomorphism

$$K_2(K) \to \bigoplus_{x \in X_0} K_1(k(x)) \oplus \bigoplus_{x \in X_\infty} K_2(K_x),$$

induced by the inclusion $K \hookrightarrow K_x$ and the boundary map $K_2(K_x) \to K_1(k(x))$ (see Def. 2 for the precise definition). Note that the residue field $k(x)$ at $x \in X_0$ is a local field, and K_x is a 2-dimensional local field in the sense of K. Kato, that is a complete discrete valuation field whose residue field is a local field. Next, a canonical continuous homomorphism $\sigma_X : \mathcal{C}_X \to \pi_1(X)^{\mathrm{ab}}$ shall be defined by local class field theory, 2-dimensional local class field theory [1] and the reciprocity law due to S. Saito [4]. Our main result is the following determination of its kernel and cokernel when the characteristic of k is 0.

Theorem 1. *Let X be a regular and geometrically connected curve over a finite extension k of $\mathbb{Q}_p$.*
(i) *The kernel of σ_X is the maximal divisible subgroup of $\mathcal{C}_X$.*
(ii) *The quotient of $\pi_1(X)^{\mathrm{ab}}$ by the topological closure $\overline{\mathrm{Im}(\sigma_X)}$ of the image of σ_X is isomorphic to $\widehat{\mathbb{Z}}^r$ with some $r \geq 0$.*

Further assume that the variety X is proper. In this case, the class group $\mathcal{C}_X$ is nothing other than $SK_1(X)$. By using this, S. Saito [4] showed the above theorem and it plays an important role in higher dimensional class field theory of K. Kato and S. Saito. The invariant r in the above theorem is called the *rank* of the compactification $\overline{X}$ of X (*op. cit.*, Def. 2.5). It depends on the type of the reduction of $\overline{X}$. In particular, we have $r = 0$ if it has potentially good reduction.

Remark. As in *op. cit.*, for a local field k with characteristic $p > 0$, the theorem above can be proved with restriction to "the prime-to-p part" in the assertion (i).

After introducing the class group of X and the reciprocity map in Section 2, we shall prove Theorem 1 in Section 3.

Throughout this paper, a *curve* over a field is an integral separated scheme of finite type over the field of dimension 1. For an abelian group A, we denote by A/n the cokernel of the map $n : A \to A$ defined by $x \mapsto nx$ for any positive integer n.

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2 Class Groups

Let X be a regular curve over a local field k with function field K, $\overline{X}$ the regular compactification of X, and X_∞ the finite set of closed points in the boundary $\overline{X} \smallsetminus X$ of X. We define a group $\mathcal{I}_X$ by

$$\mathcal{I}_X = \bigoplus_{x \in X_0} K_1(k(x)) \oplus \bigoplus_{x \in X_\infty} K_2(K_x).$$

The topology of $K_2(K_x)$ is defined in [1] (cf. *op. cit.*, I, Sect. 7). In particular, if the characteristic of k is 0, we take the discrete topologies of $K_x^\times$ and $K_2(K_x)$. The group $\mathcal{I}_X$ is endowed with the direct sum topology, that is, a subset is open if the intersection with each finite partial sum is open.

Definition. Define the *class group* $\mathcal{C}_X$ associated with X by the cokernel of the natural map $K_2(K) \to \mathcal{I}_X$ which is defined by the boundary map $K_2(K_x) \to K_1(k(x))$ and the inclusion $K \hookrightarrow K_x$. The quotient topology makes this an abelian topological group.

The *reciprocity map*

$$\sigma_X : \mathcal{C}_X \to \pi_1(X)^{\mathrm{ab}}$$

is defined as follows: For $x \in X_0$, the reciprocity map of local class field theory $K_1(k(x)) \to \pi_1(x)^{\mathrm{ab}}$ and the natural map $\pi_1(x)^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}}$ give $K_1(k(x)) \to \pi_1(X)^{\mathrm{ab}}$. For any $x \in X_\infty$, the reciprocity map of 2-dimensional local class field theory $K_2(K_x) \to \pi_1(\mathrm{Spec}(K_x))^{\mathrm{ab}}$ and the natural map $\pi_1(\mathrm{Spec}(K_x))^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}}$ define a map $K_2(K_x) \to \pi_1(X)^{\mathrm{ab}}$. Thus, we have $\mathcal{I}_X \to \pi_1(X)^{\mathrm{ab}}$. Finally, the recirprocity law of K ([4], Chap. II, Prop. 1.2) and 2-dimensional local class field theory (*op. cit.*, Chap. II, Th. 3.1) show that the homomorphism $\mathcal{I}_X \to \pi_1(X)^{\mathrm{ab}}$ defined above factors through $\mathcal{C}_X$. Thus the required homomorphism $\sigma_X : \mathcal{C}_X \to \pi_1(X)^{\mathrm{ab}}$ is obtained.

The structure map $X \to \mathrm{Spec}(k)$ induces a map $N : \mathcal{C}_X \to k^\times$ which is defined by norms over k and one denotes the kernel of this map by $V(X)$. It makes the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & V(X) \arrow[r] \arrow[d, dashed, "\tau_X"] & \mathcal{C}_X \arrow[r, "N"] \arrow[d, "\sigma_X"] & k^\times \arrow[d, "\rho_k"] \\
0 \arrow[r] & \pi_1(X)^{\mathrm{ab,geo}} \arrow[r] & \pi_1(X)^{\mathrm{ab}} \arrow[r] & \pi_1(\mathrm{Spec}(k))^{\mathrm{ab}}.
\end{tikzcd} \tag{1}$$

Here, the group $\pi_1(X)^{\mathrm{ab,geo}}$ is defined by the exactness of the lower horizontal row.

Remark. As in [6], we can define a class group and a reciprocity map for a regular *variety* over the local field k. More generally, for a regular variety over a *higher dimensional local field*, a class group may be defined as an abstract group by using Milnor K-groups of higher degree. However, there is no appropriate topology in the K-groups for degree > 2.

3 Proof of the Theorem

In this section, we shall prove Theorem 1. We denote by $\pi_1(X)^{\mathrm{ab}}_{\mathrm{cs}}$ the quotient of $\pi_1(X)^{\mathrm{ab}}$ which classifies the abelian covers of X which are completely split. The assertion (ii) is reduced to the unramified case ([4], Chap. II, Prop. 2.2, Th. 2.4) as follows:

$$\pi_1(X)^{\mathrm{ab}}/\overline{\mathrm{Im}(\sigma_X)} \simeq \pi_1(X)^{\mathrm{ab}}_{\mathrm{cs}} = \pi_1(\overline{X})^{\mathrm{ab}}_{\mathrm{cs}} \simeq \widehat{\mathbb{Z}}^r,$$

where r is the rank of $\overline{X}$.

The lemma below is used in the proof of the assertion (i) in an auxiliary role.

Lemma 2. (i) *The image of τ_X is finite.*
(ii) *The cokernel of τ_X is isomorphic to $\widehat{\mathbb{Z}}^r$.*

Proof. Since the map $N : \mathcal{C}_X \to k^\times$ is induced by norms over k, its image is finite index in $k^\times$. Thus, the commutative diagram (1) implies $\pi_1(X)^{\mathrm{ab}}/\overline{\mathrm{Im}(\sigma_X)} \simeq \pi_1(X)^{\mathrm{ab,geo}}/\overline{\mathrm{Im}(\tau_X)}$. There is an exact sequence of étale cohomology groups

$$0 \to H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \to H^1(X, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{x \in X_\infty} H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z})$$

and an isomorphism $H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z}) \simeq H^0(k(x), \mathbb{Q}/\mathbb{Z}(-1))$ of finite groups. The abelian étale fundamental group has the description $(\pi_1(X)^{\mathrm{ab}})^* \simeq H^1(X, \mathbb{Q}/\mathbb{Z})$, where the superscript "$*$" denotes the Pontrjagin dual. Thus, Theorem 1 in [7] and the above exact sequence imply the following description of $\pi_1(X)^{\mathrm{ab,geo}}$:

$$0 \to \pi_1(X)^{\mathrm{ab,geo}}_{\mathrm{tor}} \to \pi_1(X)^{\mathrm{ab,geo}} \to \widehat{\mathbb{Z}}^r \to 0,$$

where the torsion subgroup $\pi_1(X)^{\mathrm{ab,geo}}_{\mathrm{tor}}$ of $\pi_1(X)^{\mathrm{ab,geo}}$ is finite (Note that, the rank of $\overline{X}$ is the rank of the special fiber of the Néron model of the Jacobian variety of $\overline{X}$, cf. [4], Chap. II, Th. 6.2). Since the quotient group $\pi_1(X)^{\mathrm{ab,geo}}/\overline{\mathrm{Im}(\tau_X)}$ and $\pi_1(X)^{\mathrm{ab,geo}}$ are $\widehat{\mathbb{Z}}$-modules of rank r, the image of τ_X is finite. The assertions (i) and (ii) follows from it. $\square$

If we assume the following lemma, then the rest of the proof of the assertion (i) in Theorem 1 is essentially the same as in the proof of Theorem 5.1 in Chapter II of [4] (by using Lem. 2).

Lemma 3. *Let n be a positive integer. Then the map $\sigma_X : \mathcal{C}_X \to \pi_1(X)^{\mathrm{ab}}$ induces the injection*

$$\mathcal{C}_X/n \hookrightarrow \pi_1(X)^{\mathrm{ab}}/n.$$

Proof. (Compare with the proof of [4], Chap. II, Lem. 5.3.) By the duality theorem of étale cohomology groups with compact support, we have

$$\pi_1(X)^{\mathrm{ab}}/n = H^1(X, \mathbb{Z}/n)^* \simeq H^3_c(X, \mathbb{Z}/n(2)) = H^3(\overline{X}, j_!\mathbb{Z}/n(2)), \tag{2}$$

where $j : X \hookrightarrow \overline{X}$ is the open immersion. Let us consider the following diagram:

$$\begin{tikzcd} K_2(K)/n \arrow[r] \arrow[d, "h^2_n"] & \bigoplus_{x\in X_0} K_1(k(x))/n \oplus \bigoplus_{x\in X_\infty} K_2(K_x)/n \arrow[r] \arrow[d, "h"] & \mathcal{C}_X/n \arrow[r] \arrow[d, dashed] & 0 \\ H^2(K, \mathbb{Z}/n(2)) \arrow[r] & \bigoplus_{x\in \overline{X}_0} H^3_x(\overline{X}, j_!\mathbb{Z}/n(2)) \arrow[r] & H^3(\overline{X}, j_!\mathbb{Z}/n(2)). \end{tikzcd}$$

Here, the horizontal sequences are exact, and the left vertical map h^2_n is the isomorphism by the Merkur′ev-Suslin theorem [2]. The vertical map h is an isomorphism defined as follows: For $x \in X_0$, by excision and the purity theorem we have

$$H^3_x(\overline{X}, j_!\mathbb{Z}/n(2)) \simeq H^3_x(X, \mathbb{Z}/n(2)) \simeq H^1(k(x), \mathbb{Z}/n(1)).$$

Thus, Kummer theory gives an isomorphism

$$K_1(k(x))/n \xrightarrow{\sim} H^1(k(x), \mathbb{Z}/n(1)) \xrightarrow{\sim} H^3(\overline{X}, j_!\mathbb{Z}/n(2)). \tag{3}$$

For $x \in X_\infty$, let $\mathcal{O}^h_{\overline{X},x}$ be the henselization of $\mathcal{O}_{\overline{X},x}$, K^h_x the field of fractions of $\mathcal{O}^h_{\overline{X},x}$, and $j_x : \mathrm{Spec}(K^h_x) \hookrightarrow \mathrm{Spec}(\mathcal{O}^h_{\overline{X},x})$ the inclusion. By excision and Proposition 1.1 in [3], we have

$$H^3_x(\overline{X}, j_!\mathbb{Z}/n(2)) \simeq H^3_x(\mathrm{Spec}(\mathcal{O}^h_{\overline{X},x}), j_{x!}\mathbb{Z}/n(2)) \simeq H^2(K^h_x, \mathbb{Z}/n(2)).$$

The Merkur′ev-Suslin theorem gives an isomorphism

$$K_2(K_x)/n \xrightarrow{\sim} K_2(K^h_x)/n \xrightarrow{\sim} H^2(K^h_x, \mathbb{Z}/n(2)) \xrightarrow{\sim} H^3_x(\overline{X}, j_!\mathbb{Z}/n(2)).$$

By composing this and (3), the isomorphism h is defined. From the above diagram and (2), we obtain an injection

$$\mathcal{C}_X/n \hookrightarrow H^3(\overline{X}, j_!\mathbb{Z}/n(2)) \xrightarrow{\sim} \pi_1(X)^{\mathrm{ab}}/n$$

which is nothing other than the map σ_X modulo n. $\square$

Remark. Q. Tian [5] established a similar result by using relative K-groups $SK_1(\overline{X}, D)$, where $D := \overline{X} \smallsetminus X$ is the reduced Weil divisor on $\overline{X}$. However, it seems that the theorem (*op. cit.*, Th. 3.11) corresponding to the lemma above is not proved completely.

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